

NUMBER THEORY WEEK BY GANDA

University of Witwatersrand,
 October 17 - October 21, 2022
 organised by Florian Luca (florian.luca@wits.ac.za) and Fabien Pazuki (fpazuki@math.ku.dk),
 with the support of the CNRS (France),
 of the Department of Mathematical Sciences of Copenhagen University (Denmark),
 of the CoE-MaSS and of the University of Witwatersrand (South Africa).

We dedicate this conference to the 100th anniversary of the University of Witwatersrand and to the 150th anniversary of the Société Mathématique de France! Happy birthday!

PROGRAM

COURSES	Monday 17.10	Tuesday 18.10
09:00-10:30	Kevin Destagnol	Francesco Campagna
10:30-10:45	<i>Coffee break</i>	<i>Coffee break</i>
10:45-12:15	Francesco Campagna	Kevin Destagnol
12:15-13:45	<i>Lunch break</i>	<i>Lunch break</i>
13:45-15:15	Augustine Munagi	Darlison Nyirenda
15:15-15:30	<i>Coffee break</i>	<i>Coffee break</i>
15:30-17:00	Darlison Nyirenda	Augustine Munagi
17:30	<i>Social event</i>	
RESEARCH	Wednesday 19.10	
09:00-09:40	Darlison Nyirenda	
09:45-10:25	Augustine Munagi	
10:25-10:40	<i>Coffee break</i>	
10:40-11:20	Francesco Campagna	
11:20-12:00	Kevin Destagnol	
12:00-13:30	<i>Lunch break</i>	
13:30-14:10	Florian Luca	
14:10-14:25	<i>Coffee break</i>	
14:25-17:30	+STUDENT TALKS+	
18:30	<i>Conference dinner</i>	
STUDENT TALKS		Wednesday 19.10
14:25-14:50		Faith Shadow Zottor
14:55-15:20		Elizabeth Mrema
15:25-15:50		Beaullah Mugwangwavari
15:50-16:05		<i>Coffee break</i>
16:05-16:30		Faratiana Brice Razakarino
16:35-17:00		Maleka Raphadu
17:05-17:30		Atlegang Makokoe
18:30		<i>Conference dinner</i>

COURSES

TIME: Monday 17.10, 09:00-10:30.

ROOM: Fac-Science-MSB-UG-Seminar Room, UG floor, TW Kambule Mathematical Sciences Building, West Campus.

SPEAKER: **Kevin Destagnol** (Univ. Paris-Sud, kevin.destagnol@math.u-psud.fr).

TITLE: *Counting the number of varieties in a family having a rational point via sieving.*

ABSTRACT: (Part 1) A sieve is a technique for bounding the size of a set after the elements with some “undesirable properties” have been removed. The undesirable properties could be divisibility by a prime from a given set, other multiplicative constraints (divisibility by a perfect square for example) or inclusion in a set of residue classes. We will introduce in this lecture two sieves: the large sieve and the half-dimensional sieve and explain how they can be used to estimate the proportion of circles in the plane that have an integral point and the number of planar conics that admit a rational point.

TIME: Monday 17.10, 10:45-12:15.

ROOM: Fac-Science-MSB-UG-Seminar Room, UG floor, TW Kambule Mathematical Sciences Building, West Campus.

SPEAKER: **Francesco Campagna** (Univ. Hannover, campagna@math.uni-hannover.de).

TITLE: *Artin’s primitive root conjecture and the character-sum method.*

ABSTRACT: (Part 1) In 1927 Emil Artin formulated a famous conjecture concerning the (natural) density of the set of primes p for which a given integer $a \in \mathbb{Z}$ is a primitive root modulo p . However, some numerical computations soon made clear that Artin’s heuristic density was lacking a “correction factor” accounting for the possible dependence of the various local probabilities involved. In this mini-course I will review the history of Artin’s conjecture and discuss from a modern point of view the aforementioned correction factor. In particular, I will introduce the so-called *character-sum method*, a machinery developed by H. Lenstra, P. Moree and P. Stevenhagen that allows to compute conjectural densities for more general arithmetico-statistical problems à la Artin. The original primitive root problem will serve as a main example for the application of the method.

TIME: Monday 17.10, 13:45-15:15.

ROOM: Fac-Science-MSB-UG-Seminar Room, UG floor, TW Kambule Mathematical Sciences Building, West Campus.

SPEAKER: **Augustine Munagi** (Univ. of Witwatersrand, augustine.munagi@wits.ac.za).

TITLE: *Perfect Partitions: their development and recent generalizations.*

ABSTRACT: (Part 1) A perfect partition of n is a partition whose parts contain exactly one partition of every positive integer from 1 to n . Even though their inventor, P. A. MacMahon, was motivated by a problem in weights and measures, he nevertheless found immediate bijective connections to two notable combinatorial objects. In this talk we will briefly review the properties of perfect partitions, followed by an outline of their significant generalizations through the years. The most recent generalizations have, in addition, extended the domain of definition of perfect partitions to ordered and/or colored partitions, with the motivation being purely intellectual.

TIME: Monday 17.10, 15:30-17:00.

ROOM: Fac-Science-MSB-UG-Seminar Room, UG floor, TW Kambule Mathematical Sciences Building, West Campus.

SPEAKER: **Darlison Nyirenda** (Univ. of Witwatersrand, darlison.nyirenda@wits.ac.za).

TITLE: *q-Hypergeometric series, Ramanujan’s theta functions and applications.*

ABSTRACT: (Part 1) The course introduces q -Hypergeometric series and how they relate to Ramanujan’s theta functions. Transformational formulas are discussed. The original Rogers-Ramanujan identities are revisited in the spirit of a q -series approach. A few applications to combinatorics are considered, including the famous Legendre’s theorem that arises from the pentagonal numbers theorem. We further tackle the question of sums of squares in which we show how one can arrive at the Jacobi’s two squares/four squares theorems using the results within this scope. Another particular focus is Ramanujan’s partition congruences. We discuss Ramanujan’s most beautiful identity and also the modulo 7 version of it.

TIME: Tuesday 18.10, 09:00-10:30.

ROOM: Fac-Science-MSB-UG-Seminar Room, UG floor, TW Kambule Mathematical Sciences Building, West Campus.

SPEAKER: **Francesco Campagna** (Univ. Hannover, campagna@math.uni-hannover.de).

TITLE: *Artin's primitive root conjecture and the character-sum method.*

ABSTRACT: (Part 2) In 1927 Emil Artin formulated a famous conjecture concerning the (natural) density of the set of primes p for which a given integer $a \in \mathbb{Z}$ is a primitive root modulo p . However, some numerical computations soon made clear that Artin's heuristic density was lacking a "correction factor" accounting for the possible dependence of the various local probabilities involved. In this mini-course I will review the history of Artin's conjecture and discuss from a modern point of view the aforementioned correction factor. In particular, I will introduce the so-called *character-sum method*, a machinery developed by H. Lenstra, P. Moree and P. Stevenhagen that allows to compute conjectural densities for more general arithmetico-statistical problems à la Artin. The original primitive root problem will serve as a main example for the application of the method.

TIME: Tuesday 18.10, 10:45-12:15.

ROOM: Fac-Science-MSB-UG-Seminar Room, UG floor, TW Kambule Mathematical Sciences Building, West Campus.

SPEAKER: **Kevin Destagnol** (Univ. Paris-Sud, kevin.destagnol@math.u-psud.fr).

TITLE: *The Birch circle method to count rational points of bounded height.*

ABSTRACT: (Part 2) We will give an introduction in this lecture to the celebrated circle method due to Birch and explain how it can be used to estimate the number of rational points of bounded height and to prove the Hasse principle on the projective hypersurface $f(x_1, \dots, x_n) = 0$ where f is a homogeneous polynomial of degree d in $\mathbf{Z}[x_1, \dots, x_n]$ with n large enough in terms of d .

TIME: Tuesday 18.10, 13:45-15:15.

ROOM: Fac-Science-MSB-UG-Seminar Room, UG floor, TW Kambule Mathematical Sciences Building, West Campus.

SPEAKER: **Darlison Nyirenda** (Univ. of Witwatersrand, darlison.nyirenda@wits.ac.za).

TITLE: *q-Hypergeometric series, Ramanujan's theta functions and applications.*

ABSTRACT: (Part 2) The course introduces q-Hypergeometric series and how they relate to Ramanujan's theta functions. Transformational formulas are discussed. The original Rogers-Ramanujan identities are revisited in the spirit of a q-series approach. A few applications to combinatorics are considered, including the famous Legendre's theorem that arises from the pentagonal numbers theorem. We further tackle the question of sums of squares in which we show how one can arrive at the Jacobi's two squares/four squares theorems using the results within this scope. Another particular focus is Ramanujan's partition congruences. We discuss Ramanujan's most beautiful identity and also the modulo 7 version of it.

TIME: Tuesday 18.10, 15:30-17:00.

ROOM: Fac-Science-MSB-UG-Seminar Room, UG floor, TW Kambule Mathematical Sciences Building, West Campus.

SPEAKER: **Augustine Munagi** (Univ. of Witwatersrand, augustine.munagi@wits.ac.za).

TITLE: *Perfect Partitions: their development and recent generalizations.*

ABSTRACT: (Part 2) A perfect partition of n is a partition whose parts contain exactly one partition of every positive integer from 1 to n . Even though their inventor, P. A. MacMahon, was motivated by a problem in weights and measures, he nevertheless found immediate bijective connections to two notable combinatorial objects. In this talk we will briefly review the properties of perfect partitions, followed by an outline of their significant generalizations through the years. The most recent generalizations have, in addition, extended the domain of definition of perfect partitions to ordered and/or colored partitions, with the motivation being purely intellectual.

RESEARCH TALKS

TIME: Wednesday 19.10, 09:00-09:40.

ROOM: Fac-Science-MSB-UG-Seminar Room, UG floor, TW Kambule Mathematical Sciences Building, West Campus.

SPEAKER: **Augustine Munagi** (Univ. of Witwatersrand, augustine.munagi@wits.ac.za).

TITLE: *Perfect Partitions: their development and recent generalizations.*

ABSTRACT: A perfect partition of n is a partition whose parts contain exactly one partition of every positive integer from 1 to n . Even though their inventor, P. A. MacMahon, was motivated by a problem in weights and measures, he nevertheless found immediate bijective connections to two notable combinatorial objects. In this talk we will briefly review the properties of perfect partitions, followed by an outline of their significant generalizations through the years. The most recent generalizations have, in addition, extended the domain of definition of perfect partitions to ordered and/or colored partitions, with the motivation being purely intellectual.

TIME: Wednesday 19.10, 09:45-10:25.

ROOM: Fac-Science-MSB-UG-Seminar Room, UG floor, TW Kambule Mathematical Sciences Building, West Campus.

SPEAKER: **Darlison Nyirenda** (Univ. of Witwatersrand, darlison.nyirenda@wits.ac.za).

TITLE: *On parity and identities of Euler type in partitions with initial repetitions.*

ABSTRACT: Motivated by J. Fine, George Andrews considered a special class of partitions called partitions with initial repetitions. We consider subsets of Andrews' partitions with 2-initial repetitions and obtain an identity of Euler pentagonal type. We also derive several congruences modulo 2 for various partition functions in the category of partitions with initial repetitions.

TIME: Wednesday 19.10, 10:40-11:20.

ROOM: Fac-Science-MSB-UG-Seminar Room, UG floor, TW Kambule Mathematical Sciences Building, West Campus.

SPEAKER: **Francesco Campagna** (Univ. Hannover, campagna@math.uni-hannover.de).

TITLE: *Cyclic reduction densities for elliptic curves.*

ABSTRACT: For an elliptic curve E defined over a number field K , we study the density of the set $S_{E/K}$ of prime ideals \mathfrak{p} modulo which E has good reduction and the group of rational points of $E \bmod \mathfrak{p}$ is cyclic. Similarly to what happens in Artin's primitive root problem, under GRH the set $S_{E/K}$ has a density $\delta_{E/K}$ which can be described in terms of the splitting behaviour of prime ideals in a suitable family of division fields of E . In this talk I will discuss product formulas, entanglement correction factors and vanishing criteria for $\delta_{E/K}$, and here a major role is played by the character-sum method described in my mini-course. This is joint work with P. Stevenhagen.

TIME: Wednesday 19.10, 11:20-12:00.

ROOM: Fac-Science-MSB-UG-Seminar Room, UG floor, TW Kambule Mathematical Sciences Building, West Campus.

SPEAKER: **Kevin Destagnol** (Univ. of Paris-Sud, kevin.destagnol@math.u-psud.fr).

TITLE: *The Loughran-Smeets conjecture for some Châtelet type varieties.*

ABSTRACT: Serre in 1990 started a research program aiming to understand the probability that a randomly chosen diophantine equation has a solution over \mathbb{Q} . Most cases of the problem are still open today, even when the equations satisfy the Hasse principle but the Loughran-Smeets conjectures give predictions for that probability in certain cases. I will report on joint work with Efthymios Sofos regarding this problem for $x^2 - Dy^2 = P_1(t) \dots P_R(t) z^2$ where D is a square-free integer and P_i are fixed integer polynomials of any degree in n variables, with n relatively large compared to the degrees of the P_i .

TIME: Wednesday 19.10, 13:30-14:10.

ROOM: Fac-Science-MSB-UG-Seminar Room, UG floor, TW Kambule Mathematical Sciences Building, West Campus.

SPEAKER: **Florian Luca** (Univ. of Witwatersrand, florian.luca@wits.ac.za).

TITLE: *On X and Y coordinates of Pell equations in various sequences.*

ABSTRACT: In this talk, we will first survey various results concerning the presence of X or Y coordinates in various interesting sequences of integers like squares, factorials, Fibonacci numbers, rep-digits (in various bases), k -generalized Fibonacci numbers, etc. At the end we will sketch the main ideas of the proof of a recent result which

asserts that for all nonsquare integers $d > 2$, there are at most two values of Y satisfying $X^2 - dY^2 = \pm 1$ which are also Fibonacci numbers. This does not hold when $d = 2$ for which $F_1 = F_2 = 1 = Y_1$, $F_3 = 2 = Y_2$, $F_5 = 5 = Y_3$ are the Y -coordinates of the first three solutions of the corresponding Pell equation (joint with F. S. Zottor).

STUDENT TALKS

TIME: Wednesday 19.10, 14:25-14:50.

ROOM: Fac-Science-MSB-UG-Seminar Room, UG floor, TW Kambule Mathematical Sciences Building, West Campus.

SPEAKER: **Faith Shadow Zottor** (Univ. of Witwatersrand, 2076133@students.wits.ac.za).

TITLE: *On Y -coordinates of Pell equations which are Fibonacci numbers.*

ABSTRACT: Let $d \geq 2$ be an integer which is not a square. We show that if $(F_n)_{n \geq 0}$ is the Fibonacci sequence and $(X_m, Y_m)_{m \geq 1}$ is the m th solution of the Pell equation $X^2 - dY^2 = \pm 1$, then the equation $Y_m = F_n$ has at most two positive integer solutions (m, n) except for $d = 2$ when it has three solutions $(m, n) = (1, 2), (2, 3), (3, 5)$.

TIME: Wednesday 19.10, 14:55-15:20.

ROOM: Fac-Science-MSB-UG-Seminar Room, UG floor, TW Kambule Mathematical Sciences Building, West Campus.

SPEAKER: **Elizabeth Mrema** (Stellenbosch Univ., elizabethmrema1@gmail.com).

TITLE: *Minimal polynomial of a cyclotomic extension of degree 2.*

ABSTRACT: The study of cyclotomic extensions is very scarce over general fields. In this talk, we show how we can compute explicitly the minimal polynomial of a quadratic cyclotomic extension over any field. In doing so, we will develop a nice theory around primitive n^{th} roots of unity defining quadratic extensions. This is joint work with Sophie Marques.

TIME: Wednesday 19.10, 15:25-15:50.

ROOM: Fac-Science-MSB-UG-Seminar Room, UG floor, TW Kambule Mathematical Sciences Building, West Campus.

SPEAKER: **Beullah Mugwangwavari** (Univ. of Witwatersrand, 712040@students.wits.ac.za).

TITLE: *A generalization of Subbarao's finitization of Andrew's theorem.*

ABSTRACT: In 1916, P. A. MacMahon proved that for a non-negative integer n , the number of partitions of n wherein parts have multiplicity greater than 1 is equal to the number of partitions of n in which odd parts are congruent to 3 modulo 6. A few decades later G. E. Andrews proved a generalization of the theorem which was later finitized by M.V. Subbarao. We give a generalization of Subbarao's finitization of Andrews' theorem.

TIME: Wednesday 19.10, 16:05-16:30.

ROOM: Fac-Science-MSB-UG-Seminar Room, UG floor, TW Kambule Mathematical Sciences Building, West Campus.

SPEAKER: **Faratiana Brice Razakarino** (Stellenbosch Univ., brice@aims.ac.za).

TITLE: *An explicit bound on Siegel zeros of real quadratic fields.*

ABSTRACT: Let $d > 0$ be a fundamental discriminant, we consider the L -function attached to the Dirichlet character defined by the Kronecker symbol $\chi_d(n) = \left(\frac{d}{n}\right)$. We are interested in the study of the hypothetical positive real zeros of such functions that lie very close to 1, namely the Landau-Siegel zeros or simply Siegel zeros. As we did for the imaginary quadratic fields case, we follow Goldfeld-Schinzel's approach and also provide an explicit upper bound on Siegel zeros for real quadratic fields. In this talk, we first review some elementary background results and then discuss how we obtain the bound.

TIME: Wednesday 19.10, 16:35-17:00.

ROOM: Fac-Science-MSB-UG-Seminar Room, UG floor, TW Kambule Mathematical Sciences Building, West Campus.

SPEAKER: **Maleka Raphadu** (Univ. of Witwatersrand, 1407653@students.wits.ac.za).

TITLE: *Enumeration of Binary strings with applications to Compositions and Partitions.*

ABSTRACT: In this talk, we discuss some techniques for counting restricted sets of binary strings. We employ the symbolic method and recursive techniques, among others. Then we consider how these ideas facilitate the enumeration of selected classes of compositions and integer partitions.

TIME: Wednesday 19.10, 17:05-17:30.

ROOM: Fac-Science-MSB-UG-Seminar Room, UG floor, TW Kambule Mathematical Sciences Building, West Campus.

SPEAKER: **Atlegang Makokoe** (Univ. of Witwatersrand, 1877950@students.wits.ac.za).

TITLE: *Some properties and applications of Gaussian Polynomials.*

ABSTRACT: We prove properties of Gaussian polynomials, also known as q -binomial coefficients. These are natural analogues of the standard binomial coefficients. Furthermore, we demonstrate some applications of Gaussian polynomials to the theory of integer partitions and q -series.