Northcott property for special values of L-functions at integers and at 1/2

Fabien PAZUKI

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based on joint work with Jerson CARO and Riccardo PENGO

French-Korean IRN in Mathematics
Webinar in Number Theory

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Plan for the talk

1. The Northcott property.
2. Faltings and Kato.
3. Many links between L-functions and heights.
4. Heights of motives: two results.
5. Values at $1/2$.

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1. The Northcott property.

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The Northcott property

Let $\alpha \in \overline{\mathbb{Q}}$. We define its height by

$$h(\alpha) = \frac{1}{[k:\mathbb{Q}]} \sum_{v \in M_k} d_v \log \max\{1, |\alpha|_v\},$$

where M_k is the set of places v of a number field k containing α, and d_v the local degree.

The height h satisfies the *Northcott property*: for any fixed real numbers D, H, the set

$$\left\{\alpha \in \overline{\mathbb{Q}} \,|\, \deg(\alpha) \leq D, \quad h(\alpha) \leq H\right\}$$

is finite.

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Note: the numbers of height zero are exactly the roots of unity (Kronecker).

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Now a different question: pick a curve C of genus $g \geq 2$, defined over a number field k. How does one prove that $C(k)$ is finite?

We need more advanced Northcott properties.

2. Faltings and Kato.

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Height functions: the Faltings height

Let k be a number field, $\mathcal{O}_k$ its ring of integers, and A/k an abelian variety of dimension g.

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Let $p : \mathcal{A} \longrightarrow \mathcal{S} = \mathrm{Spec}(\mathcal{O}_k)$ be its Néron model, and ε be the neutral section.

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Let $\Omega^g_{\mathcal{A}/\mathcal{S}}$ be the space of $g-$differential forms and put $\omega_{A/\mathcal{S}} = \varepsilon^*\Omega^g_{\mathcal{A}/\mathcal{S}}$. It is a line bundle over $\mathrm{Spec}(\mathcal{O}_k)$.

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Let L be a line bundle over $\mathrm{Spec}(\mathcal{O}_k)$. Let s be a non-zero section of L. For every archimedean place v, fix a metric $||.||_v$. Then one defines the Arakelov degree of L to be:

$$\deg_{\mathrm{Ar}}(L) = \log \#(L/s\mathcal{O}_k) - \sum_{v \in M_k^\infty} \log ||s||_v.$$

Definition: the Faltings height of an abelian variety A/k is the Arakelov degree of the line bundle $\omega_{\mathcal{A}/\mathcal{S}}$, for the metric given by $||s||_v^2 = \frac{1}{2^g} \int_{A(\overline{k}_v)} |s \wedge \overline{s}|$:

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Fact: the Faltings height satisfies a Northcott property. This plays a crucial role in the proof of the Mordell Conjecture.

Comparing heights

Strategy: To prove that h_{Falt} satisfies a Northcott property, the strategy is to *compare* h_{Falt} with another height function (say the theta height, or any reasonable height on the moduli space of principally polarized abelian varieties) and deduce the Northcott property from this other height function.
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$$|h_{\mathrm{Falt}}(A) - 2h_{\Theta}(A, L)| \leq c(g, r) \log \max\{1, h_{\mathrm{Falt}}(A)\},$$

where (A, L) is principally polarized of dimension g with a theta structure of level r.

Height functions: the Kato height

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This Kato height measures both the complexity of the object $X \in \mathcal{MM}(F;E)$,

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Fact: *Northcott for $h_{*,\diamond}$ would imply finite generation of motivic cohomology: a motivic analogue of the Mordell-Weil Theorem.*

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Strategy: Inspired by the proof of the Northcott property for the Faltings height, we try to *compare* its generalization, the Kato height, to other interesting quantities. We do not have a theta height for motives, so we need to be creative!

Question: what kind of height function could help us measure the complexity of a motive?

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Height functions: what about L-functions?

We would like to investigate what can be said of the following function: for a mixed motive $X \in \mathcal{MM}(F; \mathbb{Q})$ defined over a number field F, the special value

$$h_n(X) := |L^*(X, n)|,$$

where n is a natural integer and, whenever the limit exists,

$$L^*(X, n) := \lim_{s \to n} \frac{L(X, s)}{(s-n)^{\mathrm{ord}_{s=n}(L(X,s))}}.$$

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Freshman's dream: yes and yes!

3. L-functions and heights.

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The many links between L-functions and heights

We will only mention four links, among the ones that are discussed in the literature:

(A) Boyd-Smyth-Deninger: Mahler measures and special values of zeta functions and Dirichlet L-functions.

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(These formulas are displayed in the appendix.)

4. Heights of motives: two first results.

Northcott property for $\zeta_K(0)$ but not for $\zeta_K(1)$

Theorem (P.-Pengo, 2024)

Let S be the set of isomorphism classes of number fields. Let ζ_F stand for the Dedekind zeta function of the field F.

(z1) *The sets $\{[F] \in S \colon |\zeta_F^*(1)| \leq B\}$ are infinite, for every big enough $B \in \mathbb{R}_{>0}$.*

(z0) *For every $B \in \mathbb{R}_{>0}$ the set $S_B := \{[F] \in S \colon |\zeta_F^*(0)| \leq B\}$ is finite. Moreover, there exist two absolute, effectively computable constants $c_1, c_2 \in \mathbb{R}_{>0}$ such that:*

$$|S_B| \leq \exp\left(c_1 B^{c_2 \log\log(B)} (\log\log(B))^3\right)$$

for every $B \in \mathbb{R}_{>3}$, where $|S_B|$ stands for the cardinal of S_B.

A part of the proof relies on the analytic class number formula:

$$\zeta_F^*(0) = -\frac{h_F}{w_F}\, R_F \qquad \zeta_F^*(1) = \frac{2^{r_1}(2\pi)^{r_2} h_F}{w_F |D_F|^{1/2}}\, R_F$$

where h_F, R_F, w_F, D_F are respectively the class number, the regulator, the number of roots of unity of F and the discriminant of F.

For $(z1)$, we may restrict our attention to F being an imaginary quadratic field, which implies that R_F is trivial. A result of Barban states that there exist $c_1, c_2 \in \mathbb{R}_{>0}$ such that:

$$\frac{1}{X} \sum_{\substack{D \in \mathcal{D} \\ |D| \leq X}} h_{\mathbb{Q}(\sqrt{-D})} = c_1 \sqrt{X}(1 + O(e^{-c_2\sqrt{\log(X)}}))$$

when $X \to +\infty$

Northcott property at the left of the critical strip

Theorem (P.-Pengo, 2024)

Let F a number field. Fix an integer $w \in \mathbb{Z}$. Let S be the set of isomorphism classes of pure motives in $\mathcal{MM}(F;\mathbb{Q})$. Then for every $B_1, B_2 \in \mathbb{R}_{\geq 0}$ and every $n \in \mathbb{Z}$ such that $n < w/2$, the set

$$\{[X] \in S,\ |L^*(X,n)| \leq B_1,\ \dim(X) \leq B_2\}$$

is finite, under the assumption that the motivic L-functions $L(X,s)$ are well defined, can be meromorphically continued to the whole complex plane, and satisfy a functional equation.

What about the center of the critical strip?

In other words, the height $h_n \colon S \to \mathbb{R}_{\geq 0}$ defined as $h_n(X) := |L^*(X, n)|$ has the Northcott property, when restricted to the subset of isomorphism classes of pure motives of weight $w > 2n$, under reasonable assumptions.

Note: The two theorems are enough to give a complete unconditional answer for $\zeta_F^*(n)$ for any fixed $n \in \mathbb{Z}$.

Question: However, it would also be interesting to understand the behaviour of the special values at the center: can one say something about the values at $s = 1/2$?

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5. Values at 1/2.

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Away from integers

For the Dedekind zeta function, we obtain a result valid for any real $\sigma \geq 1/2$.

Theorem (Caro-P.-Pengo, 2025)

Let S be the set of isomorphism classes of number fields. Let us consider the function $h_\sigma \colon S \to \mathbb{R}_{\geq 0}$ defined by

$$h_\sigma([K]) := |\zeta_K^*(\sigma)|.$$

For every $\sigma \in [\frac{1}{2}, +\infty)$ and every $B \in (\inf(h_\sigma(S)), +\infty)$ the intersection $h_\sigma(S) \cap (0, B)$ is infinite.

In other words, the height h_σ does not have the Northcott property, when $\sigma \geq 1/2$.

Elements of proof

- On the critical line, *i.e.* when $\sigma = \frac{1}{2}$, we study the second moment of the resonator coefficients constructed by Soundararajan for L-functions attached to quadratic characters.
- Inside the critical strip, but not on the critical line, *i.e.* when $\sigma \in (\frac{1}{2}, 1)$, we employ previous work of Lamzouri, concerning the similarities between the distribution of values of quadratic Dirichlet L-functions and random Euler products.
- On the boundary of the critical strip, *i.e.* when $\sigma = 1$, we use previous work of Granville and Soundararajan, which proves that special values of quadratic L-functions at $\sigma = 1$ are distributed like random Euler products.
- In the region of absolute convergence, *i.e.* when $\sigma \in (1, +\infty)$, we construct explicit families of number fields $\{K_d\}_{d\in\mathbb{N}}$ such that $[K_d : \mathbb{Q}] \to +\infty$ and $\zeta_{K_d}(\sigma) \to 1$ as $d \to +\infty$. We do so by imposing that many rational primes stay inert in each field K_d.

Thank you for your attention!

Appendix

(A) The conjectures of Boyd, which relate certain special values of L-functions to the Mahler measure of integral polynomials. The first examples are due to Smyth:

$$m(x+y+1) = \frac{3\sqrt{3}}{4\pi} L(\chi_{-3}, 2),$$

and

$$m(x+y+z+1) = \frac{7}{2\pi^2} \zeta(3).$$

(B) The formula of Gross and Zagier (1986), relating special values of L-functions of modular forms to heights of Heegner points, with the following important consequence: let E be an elliptic curve, let ω be a Néron differential, c be the Manin constant, let $K = \mathbb{Q}(\sqrt{-D})$ be such that P_K is a Heegner point on E, and $\mathcal{O}_K^*$ the units in K, then

$$L'(E/K, 1) = \frac{\|\omega\|^2}{c^2 |(\mathcal{O}_K)^*_{tors}|^2 \sqrt{D}} \widehat{h}_E(P_K).$$

(C) The conjecture of Colmez, which relates logarithmic derivatives of Artin L-functions to the Faltings height of a CM abelian variety A:

$$h_{\mathrm{Falt}}(A) \stackrel{?}{=} \sum_{\chi} m_{(E,\Phi)}(\chi) \left(\frac{L'(\chi,0)}{L(\chi,0)} + \log(f_\chi) \right),$$

where (E,Φ) is the CM-type of A and the sum runs over all the Artin characters $\chi\colon \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathbb{C}$ of conductor f_χ whose value on complex conjugation $c \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ equals $\chi(c) = -1$.

(The numbers $m_{(E,\Phi)}(\chi)$ are rational and almost all of them are zero, the traditional minus sign is included...)

(D) The strong form of the Birch and Swinnerton-Dyer conjecture, given here for an abelian variety (by Tate) over $\mathbb{Q}$:

$$L^*(A,1) \stackrel{?}{=} \frac{\Omega_\infty \prod c_p \, \#Ш(A,\mathbb{Q})}{\#A(\mathbb{Q})_{tors} \, \#\check{A}_{tors}(\mathbb{Q})} \; \mathrm{Reg}(A/\mathbb{Q}),$$

where the definition of the many terms involved will be given at the speaker's discretion.